Branching processes and Koenigs function

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Explicit solution for non-critical homogeneous in time branching processes is described.

1 Introduction

Branching processes are widely used in high energy physics, for recent physical introduction see the review[1]. For example, the well known Furry-Yule and negative binomial distributions are explained by simple branching processes with allowed transition $1 \to 2$. The use of processes with higher order transitions $1 \to n$ with n above 2 is rare due to the absence of the solution in terms of elementary functions. In this paper we describe the solution for processes with higher order transitions using recently found recursive procedure for the pure birth branching process[2]. The solution is based on the use of the Koenigs function[3] and the functional Schröder equation[4] (sometimes called the Schröder-Koenigs equation), for detailed description and extensive bibliography see [5,6]. In section 2 we describe the solution[2] for the pure birth branching process. In sections 3 and 4 respectively the procedures for non-critical branching processes and branching processes with immigration are outlined. Our conclusions are summarized in last section.

2 Solution for the general pure birth branching process

Let us recall that the branching process with continuous evolution parameter t is determined by the rates α_n for the transition of one particle into n particles with all particles evolving independently. For the pure birth branching process $\alpha_0 = 0$. The probability distribution $p_n(t)$ for the process having one particle at $t = 0$ with $p_n(0) = \delta_{1n}$ can be found from the forward Kolmogorov equation

$$\frac{\partial m}{\partial t} = f(x)\frac{\partial m}{\partial x} \qquad (1)$$

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for the probability generating function

$$m(x,t) = \sum_{n=0}^{\infty} p_n(t) x^n \tag{2}$$

with

$$f(x) = \sum_{n=2}^{\infty} \alpha_n x^n - \alpha x \tag{3}$$

and with $\alpha = \sum \alpha_n$. The Taylor expansion of the equation (1) leads to the following system of equations for the probabilities p_n

$$\frac{dp_1}{dt} = -\alpha p_1 \qquad , \tag{4}$$

$$\frac{dp_2}{dt} = \alpha_2 p_1 - 2\alpha p_2 \qquad , \tag{5}$$

$$\frac{dp_3}{dt} = \alpha_3 p_1 + 2\alpha_2 p_2 - 3\alpha p_3 \tag{6}$$

and for arbitrary n

$$\frac{dp_n}{dt} = \sum_{j=1}^{n-1} j\alpha_{n-j+1} p_j - n\alpha p_n \qquad . \tag{7}$$

Simple interpretation of the equation (7) is as follows: let us consider the state with n particles at the moment t, the change in this state is due to the arrival from states with multiplicity lower than n and to the departure to states with higher multiplicity. The arrival rate from the state with j particles is proportional to the number of particles j, to the transition rate (for one particle) to produce n-j new particles, i.e. α_{n-j+1}, and to the population density in the state j, the sum in the equation (7) goes over all states below n. The departure rate is proportional to the total transition rate (for one particle) α, to the number of particles n in this state and to the population density, this explains second term in the equation (7). Formally this system of equations is valid for any initial condition.

Let us remind the recursive solution of the equations (4)-(7) given in [2]: the probability $p_1(t)$ is $p_1 = \exp(-\alpha t)$ and

$$p_n = \sum_{j=1}^{n} \pi_{jn} p_1^j \tag{8}$$

with the following recursion for coefficients π_{jn}

$$(n-j)\pi_{jn} = \sum_{l=1}^{n-j}(n-l)b_l\pi_{j(n-l)} \qquad . \tag{9}$$

Here $b_l = \alpha_{l+1}/\alpha$ is the relative probability to produce l new particles. The recursion starts from $\pi_{11} = 1$ and the coefficient π_{nn} can be found from the initial condition

$$\pi_{nn} = -\sum_{j=1}^{n-1}\pi_{jn} \qquad . \tag{10}$$

For the case with N initial particles with $p_n^{(N)}(0) = \delta_{Nn}$ first N-1 equations are automatically valid since

$$p_1^{(N)} = p_2^{(N)} = ... = p_{N-1}^{(N)} = 0 \tag{11}$$

and the solution in this case has the following form: $p_N^{(N)} = \exp(-N\alpha t) = p_1^N$ and

$$p_n^{(N)} = \sum_{j=N}^{n}\pi_{jn}^{(N)}p_1^j \tag{12}$$

with the same recursion as (9) for the coefficients $\pi_{jn}^{(N)}$

$$(n-j)\pi_{jn}^{(N)} = \sum_{l=1}^{n-j}(n-l)b_l\pi_{j(n-l)}^{N} \qquad . \tag{13}$$

This recursion starts from $\pi_{NN}^{(N)} = 1$ and the coefficient $\pi_{nn}^{(N)}$ can be found from the relation

$$\pi_{nn}^{(N)} = -\sum_{j=N}^{n-1}\pi_{jn}^{(N)} \qquad . \tag{14}$$

One can calculate the coefficients $\pi_{jn}^{(N)}$ using the concept of the Koenigs function [5–7,2]. For the branching process starting from one particle at $t = 0$ this function is defined as a limit

$$K(x) = \lim_{n\to\infty}\frac{m(x,nt)}{p_1^n} \qquad . \tag{15}$$

The $K(x)$ has the following Taylor expansion:

$$K(x) = \sum_{j=1}^{\infty} \kappa_j x^j = \sum_{j=1}^{\infty} \pi_{1j} x^j \qquad . \tag{16}$$

For the branching process starting from N particles the Koenigs function is defined analogously:

$$K^{(N)}(x) = K^N(x) = \lim_{n\to\infty} \frac{m^N(x,nt)}{(p_1^N)^n} = \sum_{j=N}^{\infty} \kappa_j^{(N)} x^j = \sum_{j=N}^{\infty} \pi_{Nj}^{(N)} \qquad . \tag{17}$$

The recursion (13) leads to the following recurrence for the coefficients $\kappa_j^{(N)}$, $N = 1, 2, ...;\ j = N+1, N+2, ...$:

$$(j-N)\kappa_j^{(N)} = \sum_{l=1}^{j-N} (j-l) b_j \kappa_{(j-l)}^{(N)} \qquad . \tag{18}$$

Let us denote $\kappa_{x+n}^{(x)} = t_n(x)$, then $t_0(x) = 1$ and $t_n(x)$ is given by the following recursion

$$n t_n(x) = \sum_{j=1}^{n} (x+n-j) b_j t_{n-j}(x) \qquad . \tag{19}$$

It is evident from the equation (19) that the $t_n(x)$ is the polynomial of nth power in x. The $\kappa_j^{(N)}$ in terms of $t_n(x)$ is equal to $t_{j-N}(N)$.

The remarkable property of the Koenigs function is the functional Schröder equation:

$$K(m) = p_1 K(x) \qquad . \tag{20}$$

It is convenient to introduce the function $Q(x)$ inverse to the Koenigs function. Then the equation (20) gives the functional relations $m(x,t) = Q(p_1 K(x))$ and $m^N(x,t) = Q^N(p_1 K(x))$. The coefficients of the Taylor expansion for $Q^N(x) = \sum Q_j^{(N)} x^j$ can be found using the following relation (see Appendix B in [8] and references therein):

$$Q_j^{(N)} = \frac{N}{j} \kappa_{-N}^{(-j)} \qquad . \tag{21}$$

In terms of $t_n(x)$ it gives

$$Q_j^{(N)} = \frac{N}{j} t_{j-N}(-j) \qquad . \tag{22}$$

Finally, the comparison of the equation (12) with the Taylor expansion in x of the $Q^N(p_1(K(x))$ leads to the following expression for the coefficients $\pi_{jn}^{(N)}$:

$$\pi_{jn}^{(N)} = Q_j^{(N)} \kappa_n^{(j)} = \frac{N}{j} t_{j-N}(-j) t_{n-j}(j) \qquad . \tag{23}$$

3 Solution for noncritical branching processes

In this case the absorption coefficient α_0 is not equal to zero and the function $f(x)$ (3) has additional term $\alpha_0(1-x)$. Let us denote by β the positive, nearest to zero and different from 1 root of the equation $f(x) = 0$. Branching processes with $0 < \beta < 1$ are called supercritical branching processes and processes with $1 < \beta$ are called subcritical ones.

The solution for the supercritical branching processes is obtained in the following way. Let us transform m and x to m' and x' by the linear transform

$$x' = \frac{x-\beta}{1-\beta} \quad , \qquad m' = \frac{m-\beta}{1-\beta} \qquad , \tag{24}$$

this transform moves the root β to zero. Then the forward Kolmogorov equation (1) for the process with absorption transforms to the form for the pure birth branching process, therefore one can introduce the Koenigs function with

$$K\Big(\frac{m-\beta}{1-\beta}\Big) = q_1 K\Big(\frac{x-\beta}{1-\beta}\Big) \qquad , \tag{25}$$

where $q_1 = \exp(-\alpha' t)$. This leads to the expression for the $m(x,t)$

$$m(x,t) = \beta + (1-\beta) Q\Big(q_1 K\Big(\frac{x-\beta}{1-\beta}\Big)\Big) \tag{26}$$

and to the infinite series in q_1 for the probabilities p_n

$$p_n = \beta\delta_{0n} + (1-\beta) \sum_{l=n}^{\infty} \frac{(-\beta)^{l-n}}{(1-\beta)^l} \frac{l!}{n!(n-l)!} \sum_{j=n}^{l} Q_j q_1^j \kappa_l^{(j)} \qquad . \tag{27}$$

The solution for the subcritical branching processes with β above 1 can be obtained in the same way. In this case the linear transform should move 1 to zero and β to 1, i.e. $x' = (x-1)/(\beta-1)$. This leads to the similar expressions

$$m(x,t) = 1 + (\beta-1)Q\Big(q_1 K\Big(\frac{x-1}{\beta-1}\Big)\Big) \tag{28}$$

and

$$P_n = \delta_{0n} + (\beta-1)\sum_{l=n}^{\infty}\frac{(-1)^{l-n}}{(\beta-1)^l}\frac{l!}{n!(n-l)!}\sum_{j=n}^{l} Q_j q_1^j \kappa_l^{(j)} \qquad . \tag{29}$$

4 Solution for branching processes with immigration

For the branching processes with immigration there is addititonal external source of particles appearing in clusters of j particles with the differential rates β_j ($\sum \beta_j = b$). The generating function for the process starting with zero particles at $t=0$ can be written[9,10] as

$$M(x,t) = \exp\Big(\int_0^t g(m(x,\tau))d\tau\Big) \tag{30}$$

with

$$g(x) = \sum_{i=1}^{\infty} \beta_i x^i - b \qquad , \tag{31}$$

where $m(x,\tau)$ is the solution for the underlying branching process without immigration. For the underlying pure birth branching proces the equation (30) leads to the following expression

$$M(x,t) = \exp\left(-bt\right)\exp\Big(\sum_{n=1}^{\infty} v_n(t)x^n\Big) \tag{32}$$

with

$$v_n(t) = \sum_{i=1}^{n}\sum_{j=i}^{n}\pi_{jn}^{(i)}\frac{1-p_1^j(t)}{j\alpha} \qquad . \tag{33}$$

Let us denote

$$\exp\left(\sum_{n=1}^{\infty} v_n x^n\right) = 1 + \sum_{n=1}^{\infty} V_n x^n \qquad , \tag{34}$$

then the final probability $P_n(t) = \exp(-bt)V_n(t)$. The coefficients $V_n(t)$ can be calculated using the recursive relation:

$$nV_n = \sum_{j=1}^{n} j v_j V_{n-j} \qquad . \tag{35}$$

This relation is known in combinatorics and is used, for example, in the study of combitants[11–14].

5 Conclusions

Our conclusions are the following. In this paper we have derived a number of explicit expressions for the probability distributions in various branching processes. We have not derived direct expressions for the polynomials $t_n(x)$, nevertheless the given recursions can serve as a calculational tool both in theoretical and experimental studies. Mathematical formalism, described in this paper can be used in other applications, for example, one can apply the solution for the pure birth branching process to the random walk model with multiplication, when some particle jumps n steps with the rates α_n.

References

[1] R.C. Hwa, *Branching processes in multiparticle production*, in Hadronic Multiparticle Production, edited by P. Carruthers, World Scientific, Singapore, 1988.

[2] O.G. Tchikilev, *Phys.Lett.* **B471** (2000) 400; erratum, *Phys.Lett.* **B478** (2000) 459.

[3] G. Koenigs, *Recherches sur les intégrales de certaines équations fonctionnelles*, Ann. Sci. Ecole Norm. Sup. (3)1(1884), Supplément pp. 3-41; *Nouvelles recherches sur les équations fonctionnelles*, ibid. (3)2 (1885) pp. 385-404.

[4] E. Schröder, *Über unendlich viele Algorithmen zur Auflösung der Gleichungen*, Math. Ann. 2(1870) pp. 317-365; *Über iterierte Funktionen*, ibid. 3(1871) pp. 296-322.

[5] M. Kuczma, *Functional equations in a single variable*, PWN - Polish Scientific Publishers, Warszawa, 1968.

[6] M. Kuczma, B. Choczewski and R. Ger, *Iterative functional equations*, Cambridge Univ. Press, Cambridge-New York-New Rochelle- Melbourne-Sydney, 1989.

[7] G. Valiron, Fonctions analytiques, Presse Universitaire De France, Paris, 1954.

[8] J.F. Traub, *Iterative methods for the solution of equations*, Chelsea Publishing Company, New York, 1982.

[9] M.S. Bartlett, *An introduction to stochastic processes with special reference to methods and applications*, Cambridge Univ. Press, Cambridge, 1955.

[10] P.V. Chliapnikov and O.G. Tchikilev, *Phys.Lett.* **B235** (1990) 347.

[11] M. Gyulassy and S.K. Kauffmann, *Phys.Rev.Lett.* **40** (1978) 298.

[12] S.K. Kauffmann and M. Gyulassy, *J.Phys.* **A11** (1978) 1715.

[13] A.B. Balantekin and J.E. Seger *Phys.Lett.* **B266** (1991) 231.

[14] S. Hegyi, *Phys.Lett.* **B309** (1993) 443; ibid. **B318** (1993) 642.